Math 53, Discussions 116 and 118 **2022-09-30**

Limits

Questions

Question 1. As a warm-up, compute the following partial derivatives.

(a) Compute f_x and f_y for $f(x,y) = x^4 + 5xy^3$.

(b) Compute $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ if $w = \frac{e^v}{u+v^2}$.

(c) Compute $R_t(0,1)$ for $R(s,t) = te^{s/t}$.

Question 2. Draw a contour map of $f(x,y) = x^2 - y^2$ showing several level sets $f(x,y) = k$. Include the level set for $k = 0$, and at least one level set with $k > 0$ and one with $k < 0$.

Then compute f_x, f_y, f_{xx}, f_{yy} at $(x,y) = (1,1)$, and think about what your answers mean geometrically, using your contour map as guidance.

Question 3. If $f(x,y) = \frac{7}{8}(y^2+y+x)^4$, compute $f_{xyxyx}(2,0)$. Hint: it may be easier to do the derivatives with respect to x first (why are you allowed to do that?).

Question 4. Does there exist a function $f(x,y)$ with the specified partial derivatives? If so, find one. If not, explain why not.

(a) $f_x(x,y) = 2x + y, f_y(x,y) = 4y + x$

(b) $f_x(x,y) = e^x + \cos y, f_y(x,y) = x\cos y + e^y$

(c) $f_x(x,y) = e^{x^2-y^2}, f_y(x,y) = \sin(x^2)\cos(y^2)$

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1.

(a) $f_x(x, y) = 4x^3 + 5y^3$ and $f_y(x, y) = 15xy^2$.

(b) The partial derivatives are

$$\frac{\partial w}{\partial u} = \frac{e^v}{(u+v^2)^2}$$

and

$$\frac{\partial w}{\partial v} = \frac{e^v(u+v^2) + 2ve^v}{(u+v^2)^2}$$

(c) $R_t(s, t) = e^{s/t} - te^{s/t}\frac{s}{t^2}$ and then $R_t(0, 1) = 1$.

Question 2. I drew a picture in class.

Question 3. Clairaut's theorem tells us that $f_{xyxyx}(2, 0) = f_{xxxyy}(2, 0)$. Repeated differentiation gives $f_{xxxyy}(2, 0) = 42$.

Question 4.

(a) Integrating the first equation gives

$$f(x, y) = x^2 + xy + C(y)$$

where $C(y)$ is a function of y alone. Differentiating with respect to y gives

$$f_y(x, y) = x + C'(y).$$

Since $f_y(x, y) = 4y + x$ we see that $C(y) = 2y^2 + D$ for some constant D. So $f(x, y) = x^2 + xy + 2y^2$ will suffice.

(b) If such an f existed, then we would have

$$f_{xy} = \frac{\partial}{\partial y}(e^x + \cos y) = -\sin y \neq \cos y = \frac{\partial}{\partial x}(x \cos y + e^y) = f_{yx}$$

which is a violation of Clairaut's theorem. So no such f exists.

(c) Again, no such f exists, by the same reasoning as in (b).